

# The reductivity of spherical curves Part II: 4-gons

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## Abstract

The reductivity of a spherical curve represents how reduced the spherical curve is. It is unknown if there exists a spherical curve whose reductivity is four. In this paper we give an unavoidable set for spherical curves with reductivity four by considering 4-gons.

## 1 Introduction

A *spherical curve*, or a *knot projection*, is a closed curve on the 3-sphere, where each crossing is a double point and crosses transversely. In this paper we consider spherical curves with at least one crossing. We call each part of the sphere bounded by a spherical curve a *region*. In particular, we call a region which has  $n$  edges on the boundary an  *$n$ -gon*. A crossing of a spherical curve is *reducible* if there are just three regions around the crossing. A spherical curve  $P$  is reducible if  $P$  has a reducible crossing.  $P$  is *reduced* otherwise. The reductivity  $r(P)$ , defined by the second author in [6], of a spherical curve  $P$  represents how far  $P$  is from a reducible spherical curve. The precise definition of reductivity will be given in Section 2 in this paper. It is shown in [6] that the reductivity is four or less for every spherical curve although we have not found a spherical curve with reductivity four. Our next

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step is to prove that the reductivity is three or less for every spherical curve, or to find a spherical curve whose reductivity is four.

It is shown in [1] that every reduced spherical curve has a bigon (2-gon) or trigon (3-gon). This is derived from the following formula, which came from the Euler characteristic,

$$2C_2 + C_3 = 8 + C_5 + 2C_6 + 3C_7 + 4C_8 + \dots \quad (1)$$

which holds for every reduced spherical curve, where  $C_n$  is the number of the  $n$ -gons. In other words, the set  $S$  in Fig. 1 is an *unavoidable set* for

$$S = \{ \text{bigon}, \text{trigon} \} \quad T = \{ \text{trigon} \}$$

Figure 1:  $S$  is an unavoidable set for reduced spherical curves. The set  $T$  is an unavoidable set for spherical curves with reductivity three or four.

reduced spherical curves. As mentioned in [6], the reductivity of a spherical curve  $P$  is two or less if  $P$  has a bigon. Therefore we can say that the set  $T$  in Fig. 1 consisting of a trigon is an unavoidable set for spherical curves with reductivity three or four. Moreover, we can say that every spherical curve with reductivity three or four has at least eight trigons because of the formula (1).

Using the “discharging method” with the formula (1) (inspired by the story of the four color theorem), it is shown in [6] that the set  $U$  shown in Fig. 2 is also an unavoidable set for reduced spherical curves, that is, every reduced spherical curve has at least one part in  $U$ . In this paper we show

$$U = \{ \text{bigon}, \text{trigon}, \text{bigon with a diagonal}, \text{trigon with a diagonal}, \text{bigon with two diagonals} \}$$

Figure 2:  $U$  is an unavoidable set for reduced spherical curves.

the following:

**Theorem 1.1.** *If there exists a spherical curve  $P$  with reductivity four,  $P$  has at least one of the 21 parts depicted in Fig. 3, and does not have any other type of the parts in  $U$  in Fig. 2.*

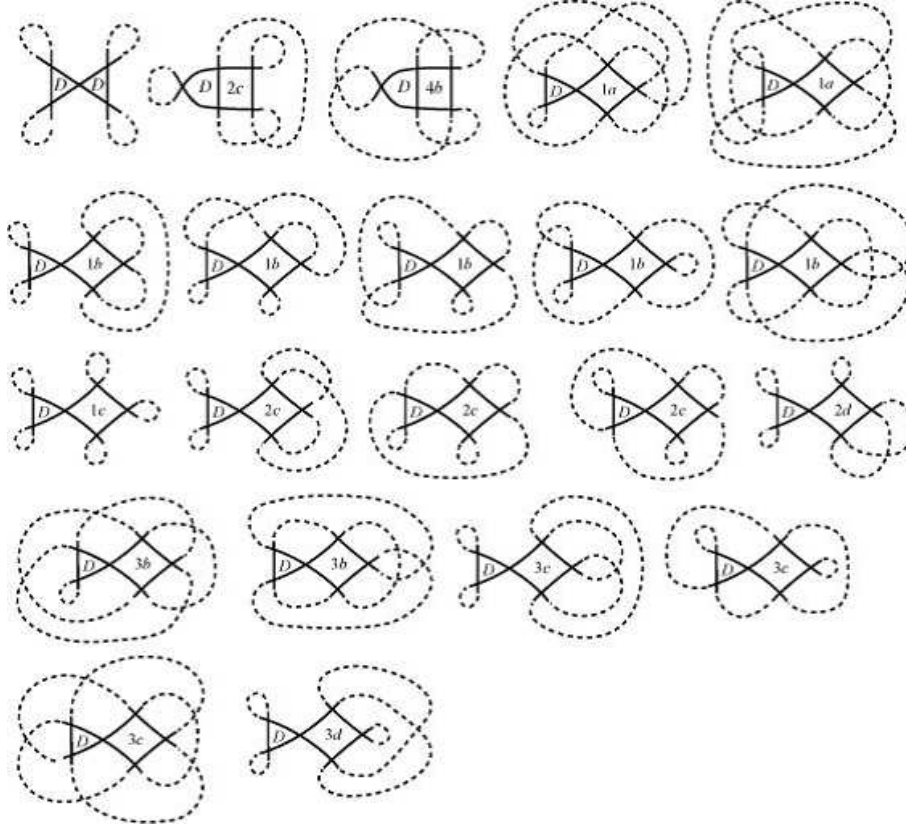


Figure 3: An unavoidable set for spherical curves with reductivity four.

Note that dotted arcs in the figures represent the outer connection in this paper. The theorem above implies that the set consisting of the 21 parts in Fig. 3 is an unavoidable set for spherical curves with reductivity four.

The rest of this paper is organized as follows: In Section 2, we review the reductivities. In Section 3, we classify 4-gons and prove Theorem 1.1.

## 2 Reductivity

In this section we review the reductivity. A *half-twisted splice* is a local transformation on a spherical curve illustrated in Fig. 4 ([2, 4]). We call  $I$  the inverse of a half-twisted splice. The *reductivity*, denoted by  $r(P)$ , of a

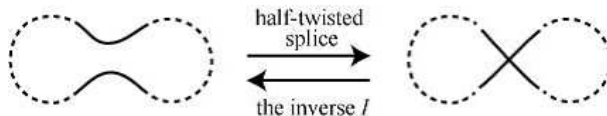


Figure 4: Half-twisted splice and its inverse  $I$ .

spherical curve  $P$  is the minimal number of  $I$  required to obtain a reducible spherical curve from  $P$ .

Considering the outer connections, bigons and trigons are divided into the two and four types as illustrated in Fig. 5. See Fig. 6 for chord diagrams, where a chord diagram of a spherical curve is the preimage of the spherical curve with two points corresponding to the same crossing connected with a segment when we consider the spherical curve as an immersion of a circle to the sphere. The following propositions are shown in [6]:

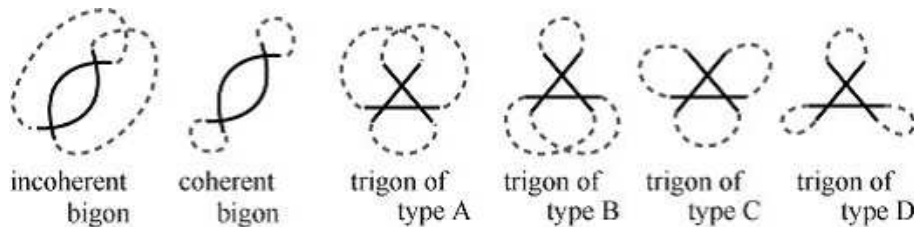


Figure 5: Bigons and trigons.

**Proposition 2.1** ([6]). *Let  $P$  be a spherical curve.*

- *If  $P$  has an incoherent bigon,  $r(P) \leq 1$ .*
- *If  $P$  has a coherent bigon,  $r(P) \leq 2$ .*
- *If  $P$  has a trigon of type A,  $r(P) \leq 2$ .*

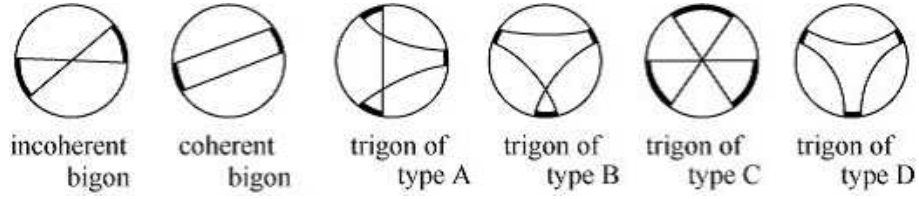


Figure 6: Bigons and trigons on chord diagrams. There are no segments on the thick arcs.

- If  $P$  has a trigon of type B,  $r(P) \leq 3$ .
- If  $P$  has a trigon of type C,  $r(P) \leq 3$ .

Remark that the converse of Proposition 2.1 does not hold. For example, the spherical curve in Fig. 7 has no bigons, but the reductivity is one. In [5], the definitions of spherical curves with reductivity one and two are

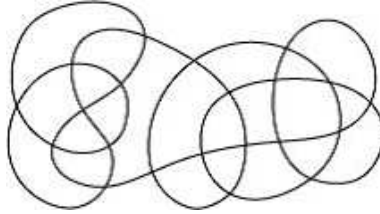


Figure 7: No bigons but reductivity one.

rephrased diagrammatically (see Fact 1 and Theorem 1 in [5]) and that is useful to determine reductivity. for example, we can say that the spherical curve  $P$  depicted in Fig. 8 has the reductivity three because  $r(P)$  is not two or less, and  $r(P) \leq 3$  because  $P$  has trigons of type B and C. We also have the following proposition:

**Proposition 2.2.** *The standard projection of a  $(3, 3n + 1)$ -torus knot has reductivity three ( $n = 1, 2, 3, \dots$ ).*

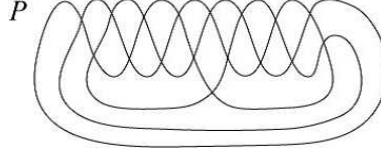


Figure 8: The reductivity of  $P$  is three.

Note that all the trigons of the standard projection of a  $(3, 3n + 1)$ -torus knot are type B. From the proposition above, we can say that there are infinitely many spherical curves with reductivity three. Similarly, we also have the following.

**Proposition 2.3.** *The standard projection of a  $(3, 3n - 1)$  and  $(4, 4n - 1)$ -torus knot has the reductivity two. The standard projection of a  $(4, 4n + 1)$ -torus knot has the reductivity three ( $n = 1, 2, 3, \dots$ ).*

Note that the standard projection of a  $(3, 3n - 1)$  and  $(4, 4n - 1)$ -torus knot has a trigon of type A.

For spherical curves  $P$  and  $Q$ , we call the spherical curve obtained from  $P$  and  $Q$  by connecting their arcs as depicted in Fig. 9 a *connected sum* of  $P$  and  $Q$ , and denote it by  $P\sharp Q$ . We say a spherical curve is *prime* if it is

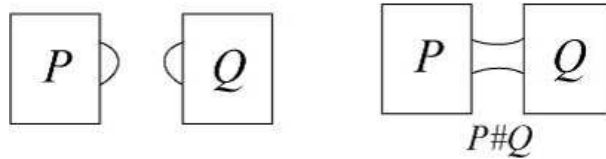


Figure 9: Connected sum.

not any connected sum of two (nontrivial) spherical curves. Since one part can be regarded as a small 1-tangle for the other one, we have the following inequality:

$$r(P\sharp Q) \leq \min\{r(P), r(Q)\}.$$

In [3], the chord diagrams for all the prime spherical curves with up to ten crossings are listed. We have the following:

**Proposition 2.4.** *All the spherical curves with 10 or less crossings except the standard projection of a  $(3, 4)$ -torus knot have the reductivity two or less.*

*Proof.* All the prime spherical curves with ten or less crossings except the standard projections of the  $(3, 4)$ ,  $(4, 3)$  and  $(3, 5)$ -torus knot (see Fig. 10 and 11) have a bigon. By Propositions 2.1, 2.2 and 2.3, the claim holds for prime spherical curves. For non-prime spherical curves, since they are connected sums of spherical curves up to nine crossings, the claim holds. Note that the reductivity of a connected sum of the standard projection of the  $(3, 4)$ -torus knot and the spherical curve with one or two crossings is zero.

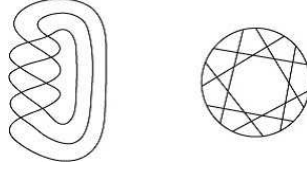


Figure 10: The standard projection of the  $(3, 4)$ -torus knot.

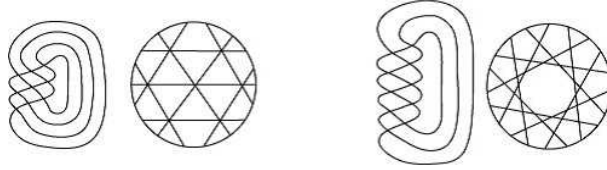


Figure 11: The standard projections of the  $(4, 3)$  and  $(3, 5)$ -torus knot.

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### 3 4-gons

In this section we classify 4-gons and prove Theorem 1.1. We first classify 4-gons into type 1 to 4 with respect to the relative orientations of the boundary

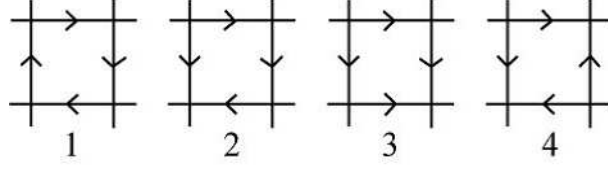


Figure 12: 4-gons with relative orientations.

as illustrated in Fig. 12. Then we classify them with respect to the outside connection and name them  $1a, 1b, 1c, \dots$  and  $4b$  as shown in Fig. 13. See Fig. 14 for chord diagrams.

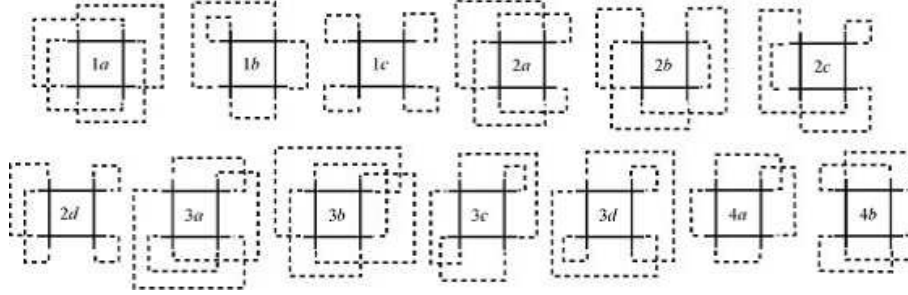


Figure 13: The 13 types of 4-gons.

We have the following:

**Proposition 3.1.** *If a spherical curve  $P$  has a 4-gon of type  $2a, 2b, 3a$  or  $4a$ , then  $r(P) \leq 3$ .*

*Proof.* The 4-gons of type  $2a, 2b, 3a$  and  $4a$  are illustrated in Fig. 15. By applying  $I$  at the marked crossing, we obtain a trigon of type A from each 4-gons.  $\square$

Now we prove Theorem 1.1.



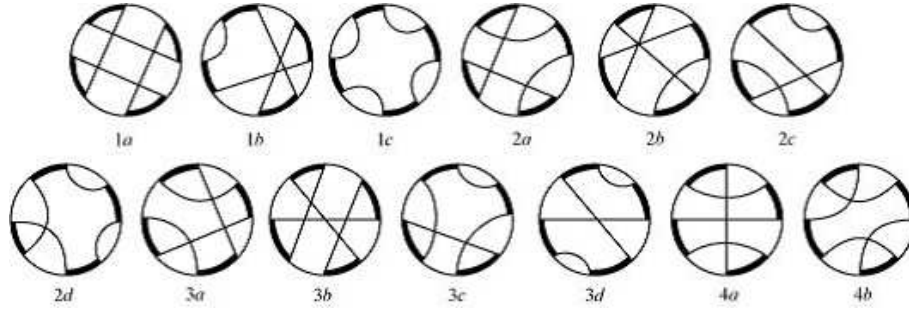


Figure 14: The 4-gons on chord diagrams. There are no segments on the thick arcs.

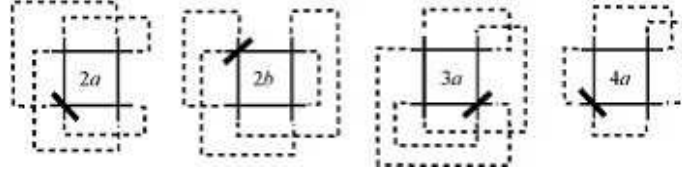


Figure 15: The 4-gons we can obtain a trigon of type A by an  $I$ .

*Proof of Theorem 1.1* If there exists a spherical curve  $P$  with  $r(P) = 4$ ,  $P$  has at least one parts of  $U$  in Fig. 2 because  $P$  is a reduced spherical curve. Here,  $P$  can not have any bigon, trigon of type A, B or C, or 4-gon of type  $2a, 2b, 3a$  or  $4a$  by Propositions 2.1 and 3.1. Then  $P$  can not have the first part of  $U$  because it is a bigon. And  $P$  can not have the second part of  $U$  because even if one of the two trigons is type D, then another one should be a trigon of type A or B because the boundary is incoherent. Hence it is sufficient to consider the rest three parts of  $U$ .

If  $P$  has the third part of  $U$ , the both two trigons should be type D, and we have just one suitable outer connection illustrated in Fig. 3.

If  $P$  has the fourth part of  $U$ , the trigon should be type D, and then the orientation of the 4-gon should be type 2 or 4 (see Fig. 16). More precisely, the 4-gon should be type  $2c, 2d$  or  $4b$  by Proposition 3.1.

For the case of  $2c$ , the thick edge of  $2c$  in Fig. 17 is to be shared with the trigon because of the orientation. Now we consider the outer connection. See Fig. 16. Considering the outer connection of D and  $2c$ ,  $\delta$  and  $\gamma$  should be connected outside. Then  $\zeta$  and  $\beta$ ,  $\eta$  and  $\varepsilon$  should be connected outside.

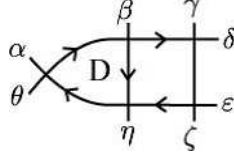


Figure 16: The fourth part of  $U$ .

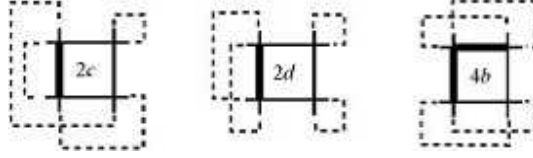


Figure 17:  $2c$ ,  $2d$  and  $4b$ .

Thus we have just one suitable outer connection of  $D$  and  $2c$ .

Next we consider the case of  $2d$ . The thick edge of  $2d$  in Fig. 17 is to be shared with the trigon. Considering the outer connection of  $D$ ,  $\alpha$  should be connected outside with  $\theta$  or  $\gamma$ . On the other hand, considering the outer connection of  $2d$ ,  $\alpha$  should be connected outside with  $\beta$ . Hence there are no suitable connection of  $D$  and  $2d$ .

Next we consider the case of  $4b$ . From a viewpoint of the orientation, each edge of  $4b$  may be shared with the trigon, and it is sufficient to consider the two thick edges in Fig. 17 by symmetry.

First, we consider the case the left-side edge of  $4b$  in Fig. 17 is shared with the trigon. Considering the outer connection of  $D$  and  $4b$ ,  $\alpha$  and  $\zeta$  should be connected outside. Then  $\gamma$  and  $\theta$ ,  $\delta$  and  $\beta$ , and  $\eta$  and  $\varepsilon$  should be connected outside. Thus we obtain just one suitable connection in this case.

Next we consider the case the upper-side edge of  $4b$  in Fig. 17 is shared with the trigon. Considering the outside connection of  $D$ ,  $\alpha$  should be connected outside with  $\theta$  or  $\zeta$ . On the other hand, considering the outer connection of  $4b$ ,  $\alpha$  should be connected outside with  $\beta$ . Hence there are no suitable connection in this case.

The parts on the fifth part of  $U$  are similarly obtained.

□

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